Uniaxial PML in Spherical and Cylindrical Coordinates for Finite-Element Time-Domain Formulations

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In order to reduce the buffer space around objects in a computational space, application of cylindrical or spherical perfectly matched layer (PML) can be very effective. Despite their wide-spread application in the finite-different time-domain (FDTD) method, we could not find any paper discussing this issue in the context of the finite-element time-domain (FETD) method. In this paper, we develop and implement uniaxial PML (UPML) for both mixed and vector wave equation (VWE) FETD formulations. In contrast to the convolutional-based formulations of the VWE FETD, we adopt the Möbius transformation technique, which is simpler in form and easier-to-implement.

***Index Terms*—Anisotropic media, finite-element time-domain (FETD), perfectly matched layer (PML).**

I. Introduction

PERFECTLY matched layer (PML) has been known as a very effective approach in truncating the computational domain in differential-based numerical techniques. It has been widely analyzed in the context of the finite-difference time-domain (FDTD) method in the electromagnetics community; however, it has witnessed less attention in finite element formulations. The proposed formulations for the finite-element time-domain (FETD) have usually considered the Cartesian formulation [1], [2] and we are not aware of any paper on either cylindrical or spherical PML in FETD. The only exception is [3], in which a 2-D conformal PML formulation based on the mixed $E-B$ FETD is implemented using the auxiliary differential equation (ADE) approach.

In this paper, we report the development and implementation of the UPML for both mixed and VWE FETD methods using the Möbius transformation technique [4], [5]. It allows efficient and straightforward discretization of the PML metrics. Furthermore, unconditional stability (US) of the VWE FETD formulation is also proven to be preserved in dispersive media when the Möbius transformation is adopted [5]. The validity of the porposed formulation is verified through a simple 2-D example in which the absolute relative error is better than 0.0032.

II. Formulation

A. PML Metrics

Consider cylindrical or spherical coordinate system with (u_1, u_2, u_3) coordinates. The PML diagonal tensor in either case can be defined as

$$\overline{\overline{\Lambda}} = \hat{u}_1 \frac{\gamma_{u_2}\gamma_{u_3}}{\gamma_{u_1}} \hat{u}_1 + \hat{u}_2 \frac{\gamma_{u_1}\gamma_{u_3}}{\gamma_{u_2}} \hat{u}_2 + \hat{u}_3 \frac{\gamma_{u_1}\gamma_{u_2}}{\gamma_{u_3}} \hat{u}_3 \tag{1}$$

in which $\gamma_{u_1}, \gamma_{u_2}, \gamma_{u_3}$ are PML metrics along each coordinate in a general form of $\gamma_{u_k}(\omega) = \alpha(u_k) + \beta(u_k)/j\omega$ [6]. The permittivity and permeability of the PML can be defined as $\overline{\overline{\varepsilon}} = \overline{\overline{\varepsilon}}_b \overline{\overline{\Lambda}}$ and $\overline{\overline{\mu}} = \overline{\overline{\mu}}_b \overline{\overline{\Lambda}}$, in which $\overline{\overline{\varepsilon}}_b$ and $\overline{\overline{\mu}}_b$ represent background material properties [1]. We discretize them in time using the Möbius transformation technique that simply involves the following substitution,

$$j\omega \mapsto \frac{2}{\Delta t} \frac{1 - z^{-1}}{1 + z^{-1}} \tag{2}$$

which yields a rational function along each direction, k, in the z-domain, as

$$\varepsilon_k(z) = \frac{c_{0_k} + c_{1_k} z^{-1} + \cdots + c_{(p_k)_k} z^{-p_k}}{1 + d_{1_k} z^{-1} + \cdots + d_{(p_k)_k} z^{-p_k}} \tag{3}$$

$$\mu_k^{-1}(z) = \frac{q_{0_k} + q_{1_k} z^{-1} + \cdots + q_{(p_k)_k} z^{-p_k}}{1 + r_{1_k} z^{-1} + \cdots + r_{(p_k)_k} z^{-p_k}} \tag{4}$$

Throughout the paper, we need to implement terms like $y(t) = \varepsilon_k(t) * x(t)$ in which $*$ represents temporal convolution. Applying the z-transformation yields $\tilde{y} = \varepsilon_k(z)\tilde{x}$. In order to implement it efficiently, we need to define the following auxiliary variables

$$\begin{aligned} \mathcal{W}^n_{\alpha,k} &= c_{\alpha_k} x^n - d_{\alpha_k} y^n + \mathcal{W}^{n-1}_{\alpha+1,k} \quad ; \quad \alpha = 1, 2, \cdots, p_k - 1 \\ \mathcal{W}^n_{\alpha,k} &= c_{\alpha_k} x^n - d_{\alpha_k} y^n \quad ; \quad \alpha = p_k \end{aligned} \tag{5}$$

by which the value of y at $t = n\Delta t$ can be obtained as

$$y^n = c_{0_k} x^n + \mathcal{W}^{n-1}_{1,k} \tag{6}$$

B. FETD Formulations

In this section, we briefly explain the first FETD formulation to be employed in this paper. The other formulation will be discussed in the long version of the paper.

Consider the VWE spatially discretized using the vector basis functions, $\boldsymbol{N}_i$, in the PML region as

$$\begin{aligned} &\mu_{u_1}^{-1} * [\mathcal{S}_{u_1}]\{e\} + \mu_{u_2}^{-1} * [\mathcal{S}_{u_2}]\{e\} \\ &\quad + \mu_{u_3}^{-1} * [\mathcal{S}_{u_3}]\{e\} + \varepsilon_{u_1} * [\mathcal{M}_{u_1}]\{\ddot{e}\} \\ &\quad + \varepsilon_{u_2} * [\mathcal{M}_{u_2}]\{\ddot{e}\} + \varepsilon_{u_3} * [\mathcal{M}_{u_3}]\{\ddot{e}\} = 0 \end{aligned} \tag{7}$$

[1]In order to avoid complexity of notation, we assume diagonal tensor for background material as well.

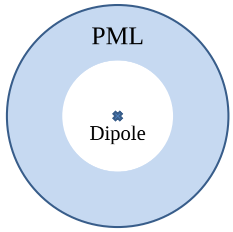

Fig. 1. A pictorial view of the problem.

where double dot indicates second temporal derivative. We assume constant material properties inside each element in the derivation of (7). Since basis functions are usually represented in Cartesian coordinates, one needs to map either basis functions to the desired coordinates or the PML tensor $\overline{\overline{\Lambda}}$ to the Cartesian coordinates. Since implementation of each off-diagonal entry requires a system of ordinary differential equations (ODEs) to be solved, we follow the former approach in this paper. The latter one destroys diagonality of the PML tensor and, therefore, produces computational overhead. In this case, mass and stiffness matrices can be obtained as

$$\mathcal{M}_k^{i,j} = \int_\Omega (\boldsymbol{N}_i \cdot \hat{u}_k)(\hat{u}_k \cdot \boldsymbol{N}_j) dV \tag{8}$$

$$\mathcal{S}_k^{i,j} = \int_\Omega ((\nabla \times \boldsymbol{N}_i) \cdot \hat{u}_k)(\hat{u}_k \cdot (\nabla \times \boldsymbol{N}_j)) dV \tag{9}$$

where $\hat{u}_k$ represents the unit vector along coordinate k. These integrals are evaluated using numerical quadrature. Having substituted convolution terms in (7) with those similar to (6) and discretized it in time using the Newmark-β method with $\beta = 1/4$, yields

$$\begin{aligned}\left\{[\mathcal{M}_t] + \frac{(\Delta t)^2}{4}[\mathcal{S}_t]\right\}\{e\}^{n+1} \\ = 2\left\{[\mathcal{M}_t] - \frac{(\Delta t)^2}{4}[\mathcal{S}_t]\right\}\{e\}^n \\ - \left\{[\mathcal{M}_t] + \frac{(\Delta t)^2}{4}[\mathcal{S}_t]\right\}\{e\}^{n-1} \\ - \left\{\{\mathcal{W}_t\}^n - 2\{\mathcal{W}_t\}^{n-1} + \{\mathcal{W}_t\}^{n-2}\right\} \\ - \frac{(\Delta t)^2}{4}\left\{\{\mathcal{G}_t\}^n + 2\{\mathcal{G}_t\}^{n-1} + \{\mathcal{G}_t\}^{n-2}\right\}\end{aligned} \tag{10}$$

where

$$[\mathcal{S}_t] = \sum_k q_{0_k}[\mathcal{S}_k], \quad [\mathcal{M}_t] = \sum_k c_{0_k}[\mathcal{M}_k] \tag{11a}$$

$$\{\mathcal{W}_t\}^n = \sum_k \{\mathcal{W}_{1,k}\}^n, \quad \{\mathcal{G}_t\}^n = \sum_k \{\mathcal{G}_{1,k}\}^n \tag{11b}$$

and where $\mathcal{W}$ and $\mathcal{G}$ are auxiliary variables for permittivity and permeability of the UPML, respectively. All auxiliary variables can be updated in a similar manner to (5).

III. Numerical Result

In this section, we provide a simple 2-D numerical example to validate the proposed formulation. Fig. 1 shows the problem consisting of two concentric circles with radii of 15 cm and 30 cm between which is filled by PML material. A magnetic dipole with the Blackman-Harris pulse shape is placed at the center of the circles and the magnetic field intensity, H_z, is recorded at the same point. The problem is solved twice: with PML and without PML (as the reference). The simulation continued for 900 time steps by which time reflected waves are not sensed (see Fig. 2). The relative error is less than 0.0032.

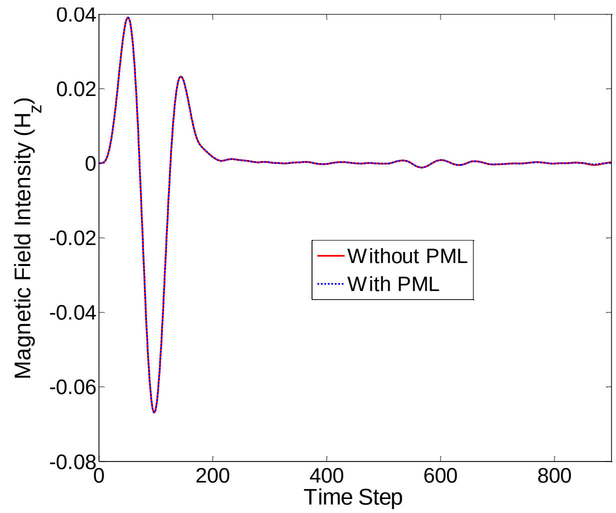

Fig. 2. Recorded magnetic field intensity during 900 time steps at the center of the problem.

IV. Conclusion

By using the Möbius transformation technique and a memory-efficient algorithm, we have proposed, developed and implemented the UMPL for both cylindrical and spherical coordinates in the FETD. Both mixed and VWE formulations of the FETD have been considered. The proposed Spherical and Cylindrical PMLs can reduce the buffer space in problems with those symmetries without increase in the computation of the PML update equations.

Full details, additional formulations, and more practical 3-D numerical results will be presented at the conference and in the long version of the paper.

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